The Expansions of Electromagnetic Fields in Cavities*

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Summary—In the theory of cavity resonators, the assumptions are frequently made that every irrotational function can be represented as the gradient of a scalar and that every divergenceless function can be represented as the rotation of a vector. These are, however, not necessarily correct. This paper corrects these misleading assumptions and describes "the theory of cavity resonators" which supplement the classical theory of Slater.

I. Introduction

EXPANDING the electromagnetic fields in terms of complete orthonormal functions, we can reduce the problem of solving Maxwell's equations in a cavity to that of determining all the expansion coefficients. In this way, Slater[1] succeeded in giving the input impedance of a cavity. However, he did not use the correct set of functions and missed a capacitance term in the input impedance expression. Furthermore he stated that for the expansion of the magnetic field $\boldsymbol{H}$ in a cavity we did not need the irrotational functions. The reason for this statement seems to be that $\boldsymbol{H}$ is itself solenoidal, namely $\nabla \cdot \boldsymbol{H}=0$. In this regard Teichmann and Wigner[2] pointed out that the component expressed as the gradient of a scalar was necessary to expand $\boldsymbol{H}$, in addition to the functions corresponding to the natural resonance modes. The contribution of this component to the input admittance is a term proportional to ω^{-1}, that is, an inductance term, where ω is the angular frequency. These functions, however, still remain incomplete in the most general case, for the assumption was made that every irrotational function can be represented as the gradient of a scalar. The function which is denoted by the symbol $\boldsymbol{G}_0$ in this paper is the missing one. In a fairly recent paper, Schelkunoff[3] made some comments on Teichmann and Wigner's work. His illustration shows that the set of natural modes is not incomplete if use is made of a short circuit which conforms to the impressed field. However, the complete sets of orthonormal functions defined in Section II of this paper are more suitable for a general discussion. These functions already have been well studied by Müller.[4] In his treatment, however, the incorrect assumption mentioned above is again taken for granted, though his final result is undoubtedly correct.

This paper describes "the theory of cavity resonators" which serves as a supplement to the classical theory of Slater. The emphasis is placed on correcting widely held assumptions that, when a function is divergenceless, we need no irrotational functions to expand the function and that, when the rotation of a function vanishes, the function can be expressed as the gradient of a scalar.

II. Complete Sets of Orthonormal Functions

For the expansion of functions defined in a closed region V, enclosed by a surface (or surfaces) S, we first have to set up appropriate complete sets of orthonormal functions in V. It is well known that the solutions of the wave equation with the boundary condition

$$\nabla^2\psi_\alpha + k_\alpha^2\psi_\alpha = 0 \quad (\text{in } V)$$
$$\psi_\alpha = 0 \quad (\text{on } S) \tag{1}$$

are capable of forming a complete set of orthonormal functions ψ_α $(\alpha=1, 2, 3, \cdots)$ which is used in expanding an arbitrary piecewise continuous scalar function in V. Similarly the solutions of

$$\nabla^2\phi_\lambda + k_\lambda^2\phi_\lambda = 0 \quad (\text{in } V)$$
$$\frac{\partial\phi_\lambda}{\partial n} = 0 \quad (\text{on } S), \tag{2}$$

where the derivative is taken in a direction normal to the surface S, are capable of forming another complete set ϕ_λ $(\lambda=0, 1, 2, \cdots)$ for the expansion of a scalar function.

For the expansion of an arbitrary piecewise continuous vector function defined in V, in like manner, we have two complete sets of orthonormal functions $\boldsymbol{\Psi}_p$ $(p=0, 1, 2, \cdots)$ and $\boldsymbol{\Phi}_q$ $(q=0, 1, 2, \cdots)$, each of which satisfies

$$\nabla^2\boldsymbol{\Psi}_p + k_p^2\boldsymbol{\Psi}_p = 0 \quad (\text{in } V)$$
$$\boldsymbol{n}\times\boldsymbol{\Psi}_p = 0, \qquad \nabla\boldsymbol{\Psi}_p = 0 \quad (\text{on } S) \tag{3}$$

and

$$\nabla^2\boldsymbol{\Phi}_q + k_q^2\boldsymbol{\Phi}_q = 0 \quad (\text{in } V)$$
$$\boldsymbol{n}\times\nabla\times\boldsymbol{\Phi}_q = 0, \qquad \boldsymbol{n}\cdot\boldsymbol{\Phi}_q = 0 \quad (\text{on } S). \tag{4}$$

It is worth noting that two boundary conditions are necessary to define these vector functions.

Some of $\boldsymbol{\Psi}_p$ and $\boldsymbol{\Phi}_q$ have the common eigenvalues k_a' $(a=1, 2, 3, \cdots)$ and are related to each other by

$$k_a\boldsymbol{\Psi}_p = \nabla\times\boldsymbol{\Phi}_q, \qquad k_a\boldsymbol{\Phi}_q = \nabla\times\boldsymbol{\Psi}_p.$$

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[2] T. Teichmann and E. P. Wigner, "Electromagnetic field expansions in loss-free cavities excited through holes," *J. Appl. Phys.*, vol. 24, pp. 262–267; March, 1953.

[3] S. A. Schelkunoff, "On representation of electromagnetic fields in cavities in terms of natural modes of oscillation," *J. Appl. Phys.*, vol. 24, pp. 262–267; March, 1953.

[4] G. Goubau, "Electromagnetische Wellenleiter und Holraume," Wissenschaftliche Verlagsgesellschaft M.B.H., Stuttgart, Germany, pp. 80–97; 1955.

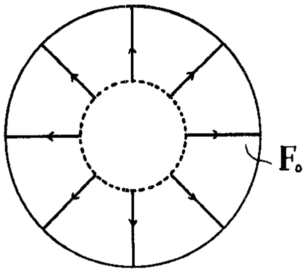


Fig. 1—F_0 functions.

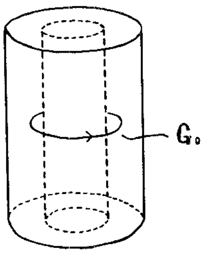


Fig. 2—G_0 functions.

Let us denote these Ψ_p and Φ_q as E_a and H_a respectively hereafter. The above relations become

$$k_a E_a = \nabla \times H_a, \qquad k_a H_a = \nabla \times E_a. \tag{5}$$

All the other Ψ_p and Φ_q will be denoted as F_α ($\alpha = 0, 1, 2, \cdots$) and G_λ ($\lambda = 0, 1, 2, \cdots$). Then it can be shown that the rotations of F_α and G_λ vanish.

$$\nabla \times F_\alpha = 0, \qquad \nabla \times G_\lambda = 0. \tag{6}$$

F_α and G_λ with nonzero eigenvalues satisfy the relations

$$k_\alpha F_\alpha = \nabla \psi_\alpha, \qquad k_\lambda G_\lambda = \nabla \phi_\lambda \tag{7}$$

where ψ_α and ϕ_λ are the functions defined by (1) and (2). If the closed region V has two or more than two separate boundaries, F_α with $k_\alpha = 0$ can exist and will be denoted by the symbol F_0. An example of such a region V is the space between two concentric spheres. F_0 satisfies

$$\nabla \times F_0 = 0, \qquad \nabla \cdot F_0 = 0 \quad (\text{in } V)$$
$$n \times F_0 = 0 \qquad (\text{on } S).$$

Therefore, by Helmholtz's theorem, F_0 can be set equal to the gradient of a scalar function ψ.

$$F_0 = \nabla \psi$$

where ψ satisfies Laplace's equation $\nabla^2 \psi = 0$ in V and the boundary condition $n \times \nabla \psi = 0$ on S. ψ can be considered as an electrostatic potential and F_0 as the electrostatic field. In a multiply connected region (the region in which there are contours which cannot be shrunk away to nothing) G_λ with $k_\lambda = 0$ can exist and will be denoted by the symbol G_0. An example of such a region is the space between two coaxial cylinders closed at both ends. In this case, G_0 corresponds to the magnetic field produced by the dc current flowing through the circuit which consists of the center conductor, the short ends, and the outer conductor. G_0 satisfies

$$\nabla \times G_0 = 0, \qquad \nabla \cdot G_0 = 0 \quad (\text{in } V)$$
$$n \cdot G_0 = 0 \qquad (\text{on } S).$$

Therefore, by Helmholtz's theorem, G_0 can be set equal to the rotation of a vector function but not to the gradient of a scalar function. If we want to set G_0 equal to the gradient of a scalar function, we have either to introduce "a cut" in the region or to use a multivalued function.

In some cases, there are many independent F_0's and G_0's; however they can all be grouped in the sets of F_α's and G_λ's, respectively, for all of them satisfy (6).

III. The Expansions of Electromagnetic Fields

We shall use the set of functions E_a and F_α in expanding the electric field, for E_a and F_α have boundary conditions similar to those of the actual field E in a cavity. For the same reason, we shall expand H in a series in terms of the H_a's and G_λ's; the current J in the E_a's and F_α's; and the charge density ρ in the ψ_α's. Thus we have

$$E = \sum_a E_a \int E \cdot E_a dv + \sum_\alpha F_\alpha \int E \cdot F_\alpha dv,$$
$$H = \sum_a H_a \int H \cdot H_a dv + \sum_\lambda G_\lambda \int H \cdot G_\lambda dv,$$
$$J = \sum_a E_a \int J \cdot E_a dv + \sum_\alpha F_\alpha \int J \cdot F_\alpha dv,$$
$$\rho = \sum_\alpha \psi_\alpha \int \rho \psi_\alpha dv. \tag{8}$$

Since $\nabla \times E$ behaves like H, the H_a's and G_λ's will be used to expand $\nabla \times E$.

$$\nabla \times E = \sum_a H_a \int \nabla \times E \cdot H_a dv + \sum_\lambda G_\lambda \int \nabla \times E \cdot G_\lambda dv. \tag{9}$$

From the vector relation

$$\nabla \cdot (E \times \nabla \times E_a) = \nabla \times E \cdot \nabla \times E_a - E \cdot \nabla \times \nabla \times E_a.$$
$$= k_a H_a \cdot \nabla \times E - k_a^2 E_a \cdot E$$

and Gauss' theorem we have

$$\int n \times E \cdot H_a dS = \int \nabla \times E \cdot H_a dv - k_a \int E \cdot E_a dv. \tag{10}$$

Similarly from

$$\nabla \cdot (E \times G_\lambda) = G_\lambda \cdot \nabla \times E - E \cdot \nabla \times G_\lambda$$
$$= G_\lambda \cdot \nabla \times E$$

we have

$$\int n \times E \cdot G_\lambda dS = \int \nabla \times E \cdot G_\lambda dv. \tag{11}$$

Inserting (10) and (11) into (9), we obtain

$$\nabla \times E = \sum_a H_a \left\{ k_a \int E \cdot E_a dv + \int n \times E \cdot H_a dS \right\},$$
$$+ \sum_\lambda G_\lambda \int n \times E \cdot G_\lambda dS. \tag{12}$$

In a corresponding way, expanding $\nabla \times H$ in a series in terms of the E_a's and F_α's and using the boundary conditions $n \times E_a = 0$ and $n \times F_\alpha = 0$ on S, we have

$$\nabla \times H = \sum_a E_a k_a \int H \cdot H_a dv. \tag{13}$$

For the expansion of $\nabla \cdot B$ we shall use the ϕ_λ's.

$$\nabla \cdot B = \sum_\lambda \phi_\lambda \int \nabla \cdot B \phi_\lambda dv. \tag{14}$$

From the relation

$$\nabla \cdot \phi_\lambda B = \phi_\lambda \nabla \cdot B + B \cdot \nabla \phi_\lambda$$
$$= \phi_\lambda \nabla \cdot B + k_\lambda B \cdot G_\lambda$$

and Gauss' theorem we have

$$\int \phi_\lambda B \cdot n dS = \int \nabla \cdot B \phi_\lambda dv + k_\lambda \int B \cdot G_\lambda dv. \tag{15}$$

Substituting in (14), we obtain

$$\nabla \cdot B = \sum_\lambda \phi_\lambda \left\{ -k_\lambda \int B \cdot G_\lambda dv + \int B \cdot n \phi_\lambda dS \right\}. \tag{16}$$

Similarly for $\nabla \cdot D$, expanding in series in the ψ_α's and using the boundary condition $\psi_\alpha = 0$ on S, we have

$$\nabla \cdot D = \sum_\alpha \psi_\alpha \left\{ -k_\alpha \int D \cdot F_\alpha dv \right\}. \tag{17}$$

We have now set up the series for the various quantities appearing in Maxwell's equations. Assuming that ϵ and μ are constant throughout in the region V, we next substitute these series in Maxwell's equations. From

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

we have

$$\sum_a H_a \left(k_a \int E \cdot E_a dv + \int n \times E \cdot H_a dS \right)$$
$$+ \sum_\lambda G_\lambda \int n \times E \cdot G_\lambda dS$$
$$+ \mu \frac{d}{dt} \left\{ \sum_a H_a \int H \cdot H_a dv + \sum_\lambda G_\lambda \int H \ G_\lambda dv \right\} = 0. \tag{18}$$

Multiplying (18) by H_a and integrating over V, on account of the orthonormal conditions, we obtain

$$k_a \int E \cdot E_a dv + \mu \frac{d}{dt} \int H \cdot H_a dv = - \int n \times E \cdot H_a dS. \tag{19}$$

Multiplying (18) by G_λ and integrating over V, we have

$$\mu \frac{d}{dt} \int H \cdot G_\lambda dv = - \int n \times E \cdot G_\lambda dS. \tag{20}$$

Similarly from

$$\nabla \times H - \frac{\partial D}{dt} = J,$$

we have

$$k_a \int H \cdot H_a dv - \epsilon \frac{d}{dt} \int E \cdot E_a dv = \int J \cdot E_a dv \tag{21}$$

$$- \epsilon \frac{d}{dt} \int E \cdot F_\alpha dv = \int J \cdot F_\alpha dv. \tag{22}$$

The equation $\nabla \cdot B = 0$ gives

$$k_\lambda \int H \cdot G_\lambda dv = \int H \cdot n \phi_\lambda dS. \tag{23}$$

Finally $\nabla \cdot D = \rho$ gives

$$-k_\alpha \epsilon \int E \cdot F_\alpha dv = \int \rho \psi_\alpha dv. \tag{24}$$

From (19)–(24), the expansion coefficients $\int E \cdot E_a dv$, $\int E \cdot F_\alpha dv$, $\int H \cdot H_a dv$, $\int H \cdot G_\lambda dv$ will be obtained. Substituting these coefficients in the first two equations in (8), the desired electric and magnetic fields will be given in the series expansion.

As shown by Slater, (22) and (24) lead to the same result if we have the equation of continuity

$$\nabla \cdot J + \frac{d}{dt} \rho = 0.$$

The only exception is the equations with the subscript $\alpha = 0$, where F_0 has no corresponding ψ_0.

To get a relation between (20) and (23) corresponding to the continuity equation, we take the time derivative of (23) and substitute in (20).

$$\mu \frac{d}{dt} \int H \cdot n \phi_\lambda dS = - k_\lambda \int n \times E \cdot G_\lambda dS$$
$$= - \int n \times E \cdot \nabla \phi_\lambda dS. \tag{25}$$

Here we introduce a differential operator ∇_S on the surface S which is equivalent to the operator $(\nabla - n(\partial/\partial n))$. Integrating

$$\nabla_S \cdot (\phi_\lambda n \times E) = \phi_\lambda \nabla_S \cdot n \times E + n \times E \cdot \nabla_S \phi_\lambda$$

over S and using the relation $\nabla \phi_\lambda = \nabla_S \phi_\lambda$ [for $(\partial \phi_\lambda / \partial n) = 0$ on S], we have

$$\oint \phi_\lambda n \times E \cdot dl$$

$$= \int \phi_\lambda \nabla_S \cdot n \times E dS + \int n \times E \cdot \nabla \phi_\lambda dS.$$

The line integral is over the perimeter of S and is equal to zero, for the surface S is a closed surface and its perimeter vanishes. Substituting in (25), we find

$$\mu \frac{d}{dt} \int H \cdot n \phi_\lambda dS = \int (\nabla_S \cdot n \times E) \phi_\lambda dS. \tag{26}$$

This is true because of

$$\nabla_S \cdot n \times E = \frac{d}{dt} B \cdot n. \tag{27}$$

Eq. (27) is the continuity for the fictitious surface magnetic charge $-B \cdot n$ and the fictitious surface magnetic current $n \times E$. It expresses the conservation of the fictitious magnetic charge. Eqs. (20) and (23) lead to the same result using (27), as in the case of (22) and (24). The only exception is the equation with the subscript $\lambda = 0$.

IV. The Input Admittance of a Cavity

We next take up a cavity with an output which couples the cavity to an outside system, and which is assumed to take the form of a waveguide (or a coaxial line). Let S_0 be the cross section of the waveguide, which forms the boundary surface between the cavity and the output (see Fig. 3). The cavity now consists of the natural cavity plus the part of the waveguide out to the surface S_0. The cavity wall S and the surface S_0 form a closed surface inside which we are solving Maxwell's equations. We assume that the tangential component of electric field, $E_\parallel$, is given on S_0 by excitation from the outside system, and we expand it in terms of the complete orthonormal modes of the waveguide.

$$E_\parallel = \sum_n E_{tn} V_n \tag{28}$$

where the E_{tn}'s are the orthonormal eigenfunctions for the transverse electric field in the waveguide and the V_n's are the expansion coefficients. V_n can be considered as a voltage associated with nth mode in the waveguide.

Next we shall expand the functions H_a and G_λ on S_0 in series in the E_{tn}'s.

$$H_a = \sum_n k \times E_{tn} \mathrm{I}_{an}, \tag{29}$$

$$G_\lambda = \sum_n k \times E_{tn} I_{\lambda n}, \tag{30}$$

where the I_{an}'s and $\mathrm{I}_{\lambda n}$'s are the expansion coefficients, k is the longitudinal unit vector of the waveguide and is equal to $-n$. From (28) and (29), we have

$$-\int_{S_0} n \times E \cdot H_a d_S = \int_{S_0} k \times E \cdot H_a d_S$$

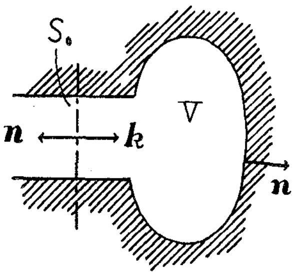


Fig. 3—Reference plane S_0.

$$= \int_{S_0} \sum_n k \times E_{tn} V_n \cdot \sum_m k \times E_{tm} I_{am} dS$$

$$= \sum_n V_n I_{an}. \tag{31}$$

Similarly from (28) and (30), we have

$$-\int_{S_0} n \times E \cdot G_\lambda dS = \sum_n V_n I_{\lambda n}. \tag{32}$$

For simplicity, we shall further assume that the angular frequency ω of the exciting field is in the vicinity of the resonant frequency $\omega_a = k_a/\sqrt{\epsilon\mu}$ of the E_a, H_a mode, which is well separated from other resonant frequencies. Then H_a component will be dominant in the magnetic field and the tangential component of E on S will be given by the approximation[5]

$$n \times E \doteqdot Z_S H \doteqdot (1 + j) \sqrt{\frac{\omega_a \mu}{2\sigma_S}} H_a \int H \cdot H_a dv \tag{33}$$

where Z_S and σ_S are the characteristic impedance and the conductivity of the wall conductor, respectively. From (33), we have

$$-\int_S n \times E \cdot H_a dS \doteqdot (1 + j) \frac{\omega_a \mu}{Q_{Sa}} \int H \cdot H_a dv, \tag{34}$$

where

$$\frac{1}{Q_{Sa}} = \frac{1}{2} \sqrt{\frac{2}{\omega_a \mu \sigma_S}} \int_S H_a^2 dS. \tag{35}$$

[5] If we take the surface roughness and irregularities of the wall into account, (34) and (35) are rewritten in the forms

$$\int_S n \times E \cdot H_a dS \doteqdot (1 + jS_a) \frac{\omega_a \mu}{Q_{Sa}} \int H \cdot H_a dv$$

$$\frac{1}{Q_{Sa}} \doteqdot \frac{F_a}{2} \sqrt{\frac{2}{\omega_a \mu_S \sigma}} \int_S H_a^2 dS$$

where F_a and $F_a S_a$ are the factors which measure the apparent increases in the surface resistance and surface reactance.

Measurements show that the relations $F_a > 1$, $S_a > 1$ usually hold. ω_a' in (39) is replaced by

$$\omega_a' = \omega_a \left(1 - \frac{S_a}{2Q_{Sa}}\right).$$

Assuming that $J=\sigma E$, we eliminate $\int E\cdot E_a dv$ from (19) and (21).

$$\epsilon\mu \frac{d^2}{dt^2}\int H\cdot H_a dv + \sigma\mu \frac{d}{dt}\int H\cdot H_a dv + k_a^2 \int H\cdot H_a dv + \left(\epsilon\frac{d}{dt}+\sigma\right)\int_S n\times E\cdot H_a dS = -\left(\epsilon\frac{d}{dt}+\sigma\right)\int_{S_0} n\times E\cdot H_a dS. \quad (36)$$

$(d/dt)=j\omega$, for we assumed the time factor $e^{j\omega t}$. If $j\omega\epsilon \gg \sigma$, as is usually the case, from (31), (34), and (36), we have

$$\int H\cdot H_a dv \left\{k_a^2 - \omega^2\epsilon\mu + j\omega\sigma\mu + (1+j)j\omega\epsilon\omega_a\mu \frac{1}{Q_{Sa}}\right\} = j\omega\epsilon \sum_n V_n I_{an} \quad (37)$$

from which we obtain

$$\int H\cdot H_a dv \doteq \frac{\sum_n V_n I_{an}/\omega_a\mu}{j\left(\frac{\omega}{\omega_a'}-\frac{\omega_a'}{\omega}\right)+\frac{1}{Q_a'}} \quad (38)$$

where

$$k_a^2 = \omega_a^2\epsilon\mu, \qquad \omega_a' = \omega_a\left(1-\frac{1}{2Q_{Sa}}\right),$$

$$\frac{1}{Q_a'} = \frac{1}{Q_a}+\frac{1}{Q_{Sa}}, \qquad \frac{1}{Q_a} = \frac{\sigma}{\omega_a\epsilon}. \quad (39)$$

Next from (20) we have

$$\mu\frac{d}{dt}\int H\cdot G_\lambda dv + \int_S n\times E\cdot G_\lambda dS = -\int_{S_0} n\times E\cdot G_\lambda dS. \quad (40)$$

We neglect the second term on the left-hand side of (40) in comparison with the first term, for $n\times E$ is sufficiently small on S. [In (36) the large terms on the left-hand side cancel each other and the surface integral $\int_S n\times E\cdot H_a dS$ cannot be neglected in the vicinity of the resonant frequency.] Inserting (30) into (40), we have

$$j\omega\mu\int H\cdot G_\lambda dv \doteq \sum_n V_n I_{\lambda n} \quad (41)$$

from which we obtain

$$\int H\cdot G_\lambda dv \doteq \frac{\sum_n V_n I_{\lambda n}}{j\omega\mu}. \quad (42)$$

Substituting (38) and (42) into (8), we have

$$H \doteq \sum_a H_a \frac{\sum_n V_n I_{an}/\omega_a\mu}{j\left(\frac{\omega}{\omega_a'}-\frac{\omega_a'}{\omega}\right)+\frac{1}{Q_a'}} + \sum_\lambda G_\lambda \frac{\sum_n V_n I_{\lambda n}}{j\omega\mu}. \quad (43)$$

This is the desired magnetic field in the form of series expansion.

For the electric field, from (21) and (22) we have, taking into account that there is no steady-state solution for $\int E\cdot F_a dv$,

$$E \doteq -j\sqrt{\frac{\mu}{\epsilon}}\sum_a E_a \frac{\sum_n V_n I_{an}/\omega\mu}{j\left(\frac{\omega}{\omega_a'}-\frac{\omega_a'}{\omega}\right)+\frac{1}{Q_a'}}. \quad (44)$$

Thus we have solved Maxwell's equations in the cavity.

The tangential magnetic field on S_0 is given by

$$H_\parallel = \sum_{a,\lambda}\sum_n\sum_m k\times E_{tm}\left\{I_{am}\frac{V_n I_{an}/\omega_a\mu}{j\left(\frac{\omega}{\omega_a'}-\frac{\omega_a'}{\omega}\right)+\frac{1}{Q_a'}} + I_{\lambda m}\frac{V_n I_{\lambda n}}{j\omega\mu}\right\}. \quad (45)$$

We are considering the cavity which has only one output with only one transmission mode. The component $H_{\parallel 1}$ of this mode is

$$H_{\parallel 1} = \sum_a k\times E_{t1}\left\{\frac{V_1 I_{a1}^2/\omega_a\mu}{j\left(\frac{\omega}{\omega_a'}-\frac{\omega_a'}{\omega}\right)+\frac{1}{Q_a'}} + \frac{V_1 I_{\lambda 1}^2}{j\omega\mu}\right\}. \quad (46)$$

On the other hand, $H_{\parallel 1}$ is expressible in the form

$$H_{\parallel 1} = k\times E_{t1} I_1 \quad (47)$$

where I_1 is the current associated with the transmission mode. From (46) and (47), we have

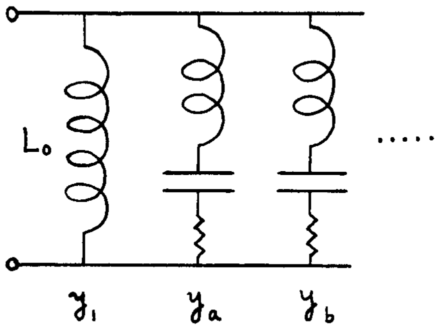
$$I_1 = \sum_a \frac{V_1 I_{a1}^2/\omega_o\mu}{j\left(\frac{\omega}{\omega_a'}-\frac{\omega_a'}{\omega}\right)+\frac{1}{Q_a'}} + \sum_\lambda \frac{V_1 I_{\lambda 1}^2}{j\omega\mu}. \quad (48)$$

The input admittance is, therefore,

$$Y = \frac{I_1}{V_1} = \sum_a \frac{I_{a1}^2/\omega_a\mu}{j\left(\frac{\omega}{\omega_a'}-\frac{\omega_a'}{\omega}\right)+\frac{1}{Q_a'}} + \sum_\lambda \frac{I_{\lambda 1}^2}{j\omega\mu}. \quad (49)$$

Similarly for cavities with two output leads, we have

$$I_1 = Y_{11}V_1 + Y_{12}V_2,$$
$$I_2 = Y_{12}V_1 + Y_{22}V_2, \quad (50)$$



$$y_1 = \frac{1}{j\omega\mu}\sum I_{\lambda 1}^2 = \frac{1}{j\omega L_0}$$

$$y_a = \frac{1}{j\left(\omega L_a - \dfrac{1}{\omega C_a}\right) + R_a}$$

where

$$L_a = \frac{\omega_a \mu}{\omega_a' I_{a1}^2}, \quad C_a = \frac{1}{\omega_a'^2 L_a}, \quad R_a = \frac{\omega_a' L_a}{Q_a'}.$$

Fig. 4—Equivalent circuit of one-entry cavity with well-separated $\omega_a's$.

where

$$Y_{ij} = \sum_a \frac{I_{ai}I_{aj}/\omega_a\mu}{j\left(\dfrac{\omega}{\omega_a'} - \dfrac{\omega_a'}{\omega}\right) + \dfrac{1}{Q_a'}} + \sum_\lambda \frac{I_{\lambda i}I_{\lambda j}}{j\omega\mu}$$

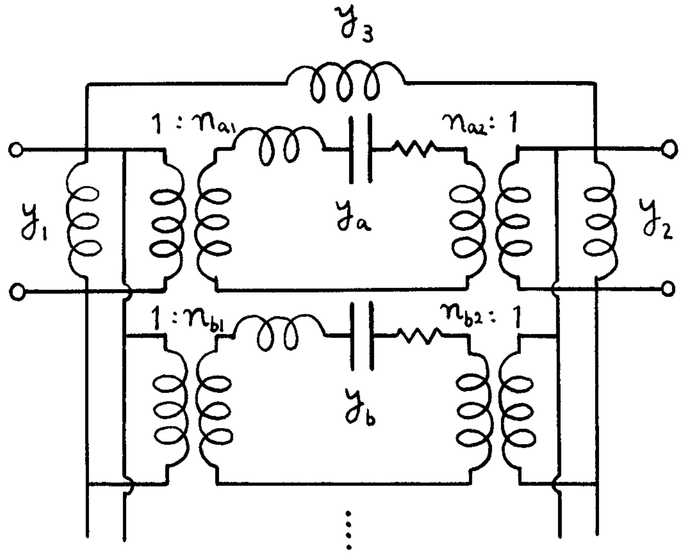
$$(i, j = 1, 2). \quad (51)$$

I_1, I_2 are the currents and V_1, V_2 are the voltages associated with the transmission modes in the output leads No. 1 and No. 2, respectively. The equivalent circuit of (49) is shown in Fig. 4 and that of (51) in Fig. 5. The generalization of the above discussion to cavities with many output leads is easy.

V. G_λ Functions

Although the relation $\nabla \cdot \boldsymbol{H} = 0$ holds in the region V, we need the G_λ functions to expand $\boldsymbol{H}$, because the condition $\nabla \cdot \boldsymbol{H} = 0$ does not necessarily mean the vanishing of the irrotational part of $\boldsymbol{H}$, as shown by Helmholtz's theorem.

We have seen that both the $\boldsymbol{H}_a$ and $\boldsymbol{G}_\lambda$ functions are necessary to set up a complete set of orthonormal functions and that an inductance term $\sum_\lambda I_{\lambda 1}^2/j\omega\mu$ from the G_λ's appears in the input admittance besides the familiar resonance terms from the $\boldsymbol{H}_a$'s. The G_λ's ($\lambda \neq 0$) are related to the conservation law of magnetic charge as explained before. If the magnetic field enters the cavity through a part of S_0 and returns into the waveguide through the other part of S_0, and if an observer does not see the outside of the cavity, he thinks that the magnetic charge $+|\boldsymbol{n}\cdot\boldsymbol{B}|$ is at the entrance of the field lines through S_0, and $-|\boldsymbol{n}\cdot\boldsymbol{B}_1$ is at their exit from S_0, and that the magnetic current $\boldsymbol{n}\times\boldsymbol{E}$ flows between them. Such a magnetic field cannot be expressed by the $\boldsymbol{H}_a$'s alone but requires the G_λ's as



$$y_1 = \frac{1}{j\omega\mu}\sum (I_{\lambda 1}^2 + I_{\lambda 1}I_{\lambda 2}), \quad y_2 = \frac{1}{j\omega\mu}\sum (I_{\lambda 2}^2 + I_{\lambda 1}I_{\lambda 2})$$

$$y_3 = -\frac{1}{j\omega\mu}\sum I_{\lambda 1}I_{\lambda 2},$$

$$1{:}n_{a1} = 1{:}n\sqrt{\frac{I_{a1}}{I_{a2}}}, \quad 1{:}n_{a2} = 1{:}-n\sqrt{\frac{I_{a2}}{I_{a1}}}$$

$$y_a = \frac{1}{j\left(\omega L_a - \dfrac{1}{\omega C_a}\right) + R_a}$$

where

$$L_a = n^2\frac{\omega_a\mu}{\omega_a' I_{a1}I_{a2}}, \quad C_a = \frac{1}{\omega_a'^2 L_a}, \quad R_a = \frac{\omega_a' L_a}{Q_a'}.$$

Fig. 5—Equivalent circuit of two-entry cavity with well-separated $\omega_a's$.

well. If we neglect the G_λ's, we have to assume that there is no magnetic field through S_0.

For illustration, let us consider a cavity with the smallest ω_a satisfying $\omega_a \gg \omega$. The input admittance can now be written in the form

$$Y = \sum_a \frac{I_{a1}^2/\omega_a\mu}{j\left(\dfrac{\omega}{\omega_a'} - \dfrac{\omega_a'}{\omega}\right) + \dfrac{1}{Q_a'}} + \sum_\lambda \frac{I_{\lambda 1}^2}{j\omega\mu}$$

$$\doteqdot j\omega C_0 + \frac{1}{j\omega L_0} \quad (52)$$

where

$$C_0 = \sum_a \frac{I_{a1}^2}{\omega_a^2\mu}, \quad \frac{1}{L_0} = \sum_\lambda \frac{I_{\lambda 1}^2}{\mu}. \quad (53)$$

If the G_λ's were neglected, the inductive term $1/j\omega L_0$ would not appear and we might come to the conclusion that the input admittance must be always capacitive when $\omega_a \gg \omega$ $(a = 1, 2, 3, \cdots)$. The input admittance can be both inductive and capacitive depending on the magnitudes of C_0 and L_0. If $\omega C_0 = 1/\omega L_0$, the cavity shows a resonance which is different from the resonance of $\boldsymbol{E}_a$ and $\boldsymbol{H}_a$. A length of waveguide short-circuited at one end and shunted by a window at the other end can be considered as a cavity. S_0 may be placed just in front of the window. If the length of the cavity is $\lambda_g/4$, certainly we have $\omega_a \gg \omega$. The input admittance of the cavity is, in this case, the shunt admittance of the window and it can be inductive, capacitive, or resonant, depending on the type of the window. This shows that we need the term $1/j\omega L_0$ in (52) and hence the $\boldsymbol{G}_\lambda$ functions to expand $\boldsymbol{H}$.

In a certain type of cavities, we need the $\boldsymbol{G}_0$ function as well. Though the fictitious magnetic current may close upon itself and there may be no fictitious magnetic charge on S_0, this function can contribute an inductance term to the input admittance. The cavity with a coupling loop and a coaxial line output lead is an example (Fig. 6). S_0 may be placed in the coaxial line some distance away from the coupling loop. In this case, we have the $\boldsymbol{G}_0$ function corresponding to the magnetic field produced by the dc circuit which consists of S_0, the outer and inner conductors of the coaxial line and the loop. If the TEM wave is the only transmission mode in the output lead, the normal component of $\boldsymbol{H}$ on S_0 is vanishingly small. Still, we have an inductance term in the input admittance, for $\boldsymbol{n}\times\boldsymbol{E}$ on S_0 can induce the $\boldsymbol{G}_0$ component of $\boldsymbol{H}$ in the cavity. The necessity for the inductance term is easily seen, if we consider the admittance at a very low frequency.

Every $\boldsymbol{H}_a$ has a corresponding $\boldsymbol{E}_a$, but $\boldsymbol{G}_0$ has not. Hence, $\boldsymbol{G}_0$ is not a resonance mode. $\boldsymbol{G}_0$ has no relation to the fictitious magnetic charge on S_0 and in this respect $\boldsymbol{G}_0$ is distinct from all the other $\boldsymbol{G}_\lambda$'s.

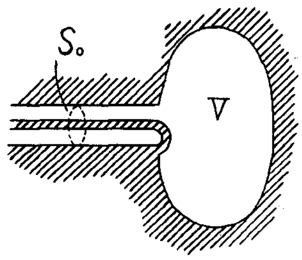


Fig. 6—Cavity with loop.

VI. Examples

To justify our conclusion about the $\boldsymbol{G}_\lambda$ functions, we shall calculate the input admittance of simple cavities. First, the input admittance of a short-circuited rectangular waveguide ($l_x \times l_y$) will be taken up. The reference plane S_0 is l_z away from the shorted end. This example was discussed in detail in Schelkunoff's paper,[3] still it might be instructive to follow our steps in the same example.

Considering the waveguide from the shorted end up to S_0 as a cavity, we have two types of $\boldsymbol{H}_a$ functions.

$$\boldsymbol{H}_a = -\boldsymbol{i}A\left(\frac{\pi m_y}{l_y k_m}\right)\sin\frac{\pi m_x x}{l_x}\cos\frac{\pi m_y y}{l_y}\cos\frac{\pi m_z z}{l_z} + \boldsymbol{j}A\left(\frac{\pi m_x}{l_x k_m}\right)\cos\frac{\pi m_x x}{l_x}\sin\frac{\pi m_y y}{l_y}\cos\frac{\pi m_z z}{l_z} \tag{54}$$

$$\boldsymbol{H}_a = -\boldsymbol{i}A\left(\frac{\pi m_z}{l_z k_m}\right)\left(\frac{\pi m_x}{l_x k_m}\right)\sin\frac{\pi m_x x}{l_x}\cos\frac{\pi m_y y}{l_y}\cos\frac{\pi m_z z}{l_z} - \boldsymbol{j}A\left(\frac{\pi m_z}{l_z k_m}\right)\left(\frac{\pi m_y}{l_y k_m}\right)\cos\frac{\pi m_x x}{l_x}\sin\frac{\pi m_y y}{l_y}\cos\frac{\pi m_z z}{l_z} + \boldsymbol{k}A\frac{1}{k_m^2}\left[\left(\frac{\pi m_x}{l_x}\right)^2+\left(\frac{\pi m_y}{l_y}\right)^2\right]\cdot\cos\frac{\pi m_x x}{l_x}\cos\frac{\pi m_y y}{l_y}\sin\frac{\pi m_z z}{l_z} \tag{55}$$

where

$$k_m^2 = \left(\frac{\pi m_x}{l_x}\right)^2+\left(\frac{\pi m_y}{l_y}\right)^2+\left(\frac{\pi m_z}{l_z}\right)^2=\left(\frac{\omega_a}{c}\right)^2 \tag{56}$$

$$A = \sqrt{\frac{\epsilon_{mx}\epsilon_{my}\epsilon_{mz}}{l_x l_y l_z}}\,\frac{k_m}{\sqrt{\left(\frac{\pi m_x}{l_x}\right)^2+\left(\frac{\pi m_y}{l_y}\right)^2}}. \tag{57}$$

The $\boldsymbol{G}_\lambda$ functions are

$$\boldsymbol{G}_\lambda = \sqrt{\frac{\epsilon_{mx}\epsilon_{my}\epsilon_{mz}}{l_x l_y l_z}}\,\frac{1}{k_\lambda}\left\{\boldsymbol{i}\left(\frac{\pi m_x}{l_x}\right)\sin\frac{\pi m_x x}{l_y}\cos\frac{\pi m_y y}{l_y}\cos\frac{\pi m_z z}{l_z} + \boldsymbol{j}\left(\frac{\pi m_y}{l_y}\right)\cos\frac{\pi m_x x}{l_x}\sin\frac{\pi m_y y}{l_x}\cos\frac{\pi m_z z}{l_z} + \boldsymbol{k}\left(\frac{\pi m_z}{l_z}\right)\cos\frac{\pi m_x x}{l_x}\cos\frac{\pi m_y y}{l_y}\sin\frac{\pi m_z z}{l_z}\right\}, \tag{58}$$

where

$$k_\lambda^2 = \left(\frac{\pi m_x}{l_x}\right)^2+\left(\frac{\pi m_y}{l_y}\right)^2+\left(\frac{\pi m_z}{l_z}\right)^2. \tag{59}$$

The ϵ_m's are the Neumann factors. The cavity is excited by the TE_{10} mode in the waveguide. The tangential component $\boldsymbol{E}_{t1}$ of this mode is

$$\boldsymbol{E}_{t1} = \boldsymbol{j}\sqrt{\frac{2}{l_x l_y}}\sin\frac{\pi x}{l_x} \tag{60}$$

where $\sqrt{2/l_x l_y}$ is the normalizing factor. In this case, we need $\boldsymbol{H}_a$ with $m_x=1$, $m_y=0$, $m_z=1, 2, 3, \cdots$ in (55) and $\boldsymbol{G}_\lambda$ with $m_x=1$, $m_y=0$, $m_z=0, 1, 2, \cdots$ only. These are the functions used by Schelkunoff in his second approach. From (29) and (30), we obtain

$$I_{a1} = \sqrt{\frac{2}{l_z}} \frac{\pi m_z}{l_z k_m} \cos m_z \pi, \tag{61}$$

$$I_{\lambda 1} = -\frac{1}{k_\lambda} \sqrt{\frac{\epsilon m_z}{l_z}} \frac{\pi}{l_x} \cos m_z \pi. \tag{62}$$

Hence

$$I_{a1}^2 = \frac{2}{l_z} \left(\frac{\pi m_z}{l_z k_m} \right)^2$$

$$\sum I_{\lambda 1}^2 = \sum_{m_z=0}^{\infty} \frac{1}{k_\lambda^2} \frac{\epsilon_{m_z}}{l_z} \left(\frac{\pi}{l_x} \right)^2$$

$$= \frac{1}{l_x} \left\{ 1 + 2 \sum_{m_z=1}^{\infty} \frac{\left(\frac{\pi}{l_x} \right)^2}{\left(\frac{\pi}{l_x} \right)^2 + \left(\frac{\pi m_z}{l_z} \right)^2} \right\}.$$

Inserting these values in (49) and neglecting the effect of losses, we have

$$Y = \frac{1}{j\mu} \frac{1}{l_z} \left[\sum_a \frac{2\omega \left(\frac{\pi m_z}{l_z} \right)^2 \Big/ k_m^2}{\omega^2 - \omega_a^2} + \frac{1}{\omega} \left\{ 1 + 2 \sum_{m_z=1}^{\infty} \frac{\left(\frac{\pi}{l_x} \right)^2}{\left(\frac{\pi}{l_x} \right)^2 + \left(\frac{\pi m_z}{l_z} \right)^2} \right\} \right]. \tag{63}$$

On the other hand, from transmission line theory, we have

$$Y = \sqrt{\frac{\epsilon}{\mu}} \frac{\sqrt{\omega^2 - \omega_c^2}}{j\omega} \cot \beta l_z \tag{64}$$

where

$$\beta = \sqrt{\epsilon\mu} \sqrt{\omega^2 - \omega_c^2}$$

$$\omega_c^2 = \left(\frac{\pi}{l_x} \right)^2 c^2 = \left(\frac{\pi}{l_x} \right)^2 \Big/ \epsilon\mu.$$

Using the relation

$$\cot \theta = \frac{1}{\theta} \left\{ 1 + \sum_{n=1}^{\infty} \frac{2}{1 - \left(\frac{n\pi}{\theta} \right)^2} \right\}$$

(64) is rewritten in the form

$$Y = \frac{1}{j\omega\mu l_z} \left| 1 + \sum_{n=1}^{\infty} \frac{2 \left\{ \omega^2 - \left(\frac{\pi}{l_x} \right)^2 c^2 \right\}}{\omega^2 - c^2 \left\{ \left(\frac{\pi}{l_x} \right)^2 + \left(\frac{n\pi}{l_z} \right)^2 \right\}} \right|. \tag{65}$$

Eqs. (63) and (65) can be shown to be identical by a little algebra, leading to the conclusion that (63) is equal to (64). Eqs. (63) and (64) represent the same admittance and hence this conclusion is quite reasonable. However, if we neglect the $\boldsymbol{G}_\lambda$ functions from the beginning, the ω^{-1} term will not appear in (63) and the two admittances differ from each other.

Next, we shall take up a short-circuited coaxial line, showing the contribution of the $\boldsymbol{G}_0$ function. The distance between the reference plane S_0 and the short end is L. The tangential component $\boldsymbol{E}_{t1}$ of the exciting field is

$$\boldsymbol{E}_{t1} = \boldsymbol{i}_r \frac{1}{\sqrt{2\pi \log b/a}} \frac{1}{r} \tag{66}$$

where a is the radius of inner conductor and b is the inner radius of outer conductor. The induced $\boldsymbol{H}_a$'s by this field are

$$\boldsymbol{H}_a = \boldsymbol{i}_\phi \frac{1}{\sqrt{2\pi \log b/a}} \sqrt{\frac{2}{L}} \frac{1}{r} \cos \frac{n\pi z}{L} (n = 1, 2, 3, \cdots) \tag{67}$$

and the $\boldsymbol{G}_0$ is

$$\boldsymbol{G}_0 = \boldsymbol{i}_\phi \frac{1}{\sqrt{2\pi \log b/a}} \sqrt{\frac{1}{L}} \frac{1}{r}. \tag{68}$$

From (29) and (30), we obtain

$$I_{a1} = \sqrt{\frac{2}{L}} \cos n\pi$$

$$I_{\lambda 1} = \sqrt{\frac{1}{L}}.$$

Using the relation

$$\omega_a = \frac{n\pi}{L} \frac{1}{\sqrt{\epsilon\mu}}$$

and (49), we have a well known expression of the input admittance.

$$Y = \sum_a \frac{\frac{2}{L} \Big/ \omega_a \mu}{j \left(\frac{\omega}{\omega_a} - \frac{\omega_a}{\omega} \right)} + \frac{\frac{1}{L}}{j\omega\mu}$$

$$= \frac{1}{j\omega\mu L} \left\{ 1 + \sum_{n=1}^{\infty} \frac{2}{1 - \left(\frac{n\pi}{\omega L \sqrt{\epsilon\mu}} \right)^2} \right\}$$

$$= -j \sqrt{\frac{\epsilon}{\mu}} \cot \frac{2\pi}{\lambda_g} L. \tag{69}$$

If we neglect the $\boldsymbol{G}_0$ function, we have again a faulty result in this case.

These examples show that we need the G_λ's as well as the H_a's to expand the magnetic field, though $\nabla \cdot H = 0$ throughout in the cavity.

VII. Appendix

Helmholtz's Theorem

Let V be a region inside a closed surface (or surfaces) S. Then any vector function F defined in V can be expressed as the sum of the gradient of a scalar and the rotation of a vector.

$$F(r) = \nabla\left\{\oint_S \frac{F(r')}{4\pi R}\cdot n dS' - \int_V \frac{\nabla'\cdot F(r')}{4\pi R} dV'\right\} + \nabla \times \left\{\oint_S \frac{F(r')}{4\pi R} \times n dS' + \int_V \frac{\nabla' \times F(r')}{4\pi R} dV'\right\}^{6} \quad (70)$$

where

$$R = |r - r'|,$$

∇' is a differential operator on r', and n is the outer normal on S. Eq. (70) shows that if we have two conditions $\nabla \cdot F = 0$ in V and $n \cdot F = 0$ on S, then F can be set equal to the rotation of a vector. If we have two conditions $\nabla \times F = 0$ in V and $n \times F = 0$ on S, then F can be set equal to the gradient of a scalar. $\nabla \times F = 0$ in V does not necessarily mean that $F = \nabla\phi$ where ϕ is a scalar function of position. In a corresponding way $\nabla \cdot F = 0$ in V does not necessarily mean $F = \nabla \times A$ where A is a vector function of position. The examples are the G_0 and F_0 functions defined in Section II.

Completeness of $\boldsymbol{\Psi}$ *and* $\boldsymbol{\Phi}$

Consider the relevance

$$k^2 \int \Psi^2 dv = \int \{(\nabla \times \Psi)^2 + (\nabla \cdot \Psi)^2\} dv - 2\int n \times \Psi \cdot \nabla \times \Psi dS \quad (71)$$

and take the variation. A little manipulation shows that

$$\delta k^2 \int \Psi^2 dv = -2\int \delta\Psi \cdot (\nabla^2\Psi + k^2\Psi) dv + 2\int \{\Psi \times n \cdot \nabla \times \delta\Psi + \nabla \cdot \Psi n \cdot \delta\Psi\} dS. \quad (72)$$

Therefore

$$k^2 = \frac{\int \{(\nabla\times\Psi)^2 + (\nabla\cdot\Psi)^2\} dv - 2\int n + \Psi \cdot \nabla \times \Psi dS}{\int \Psi^2 dv} \quad (73)$$

is the appropriate variational expression for the eigenvalue of Ψ. We can choose Ψ_p one by one, each of which makes

$$\Omega = \int \{(\nabla \times \Psi_p)^2 + (\nabla \cdot \Psi_p)^2\} dv + 2\int \Psi_p \times n \cdot \nabla \times \Psi_p dS \quad (74)$$

a minimum under the normalizing condition and the orthogonal conditions.

$$\int \Psi_p{}^2 dv = 1, \qquad \int \Psi_p \cdot \Psi_q dv = 0 \qquad q < p. \quad (75)$$

An infinite set of functions thus selected forms a complete set of orthonormal functions as the conventional proof shows.[7] Every Ψ_p satisfies (3). Hence, we come to the conclusion that the solutions of (3) are capable of forming a complete set of orthonormal functions.

The variational expression for the eigenvalue of Φ is

$$k^2 = \frac{\int \{(\nabla\times\Phi)^2 + (\nabla\cdot\Phi)^2\} dv - 2\int (n \cdot \Phi)(\nabla \cdot \Phi) dS}{\int \Phi^2 dv} \cdot \quad (76)$$

In a similar way as for Ψ, we easily verify that the solutions of (4) are capable of forming another complete set of orthonormal functions.

With the aid of Helmholtz's theorem, it can be shown that the Ψ_p's are divided into two groups, the E_a's and the F_α's and the Φ_q's into the H_a's and the G_λ's.

Eq. (72) and $k_0{}^2 = 0$ shows that F_0 satisfies $\nabla \times F_0 = 0$ and $\nabla \cdot F_0 = 0$ in V since $n \times \Psi = 0$ on S. Similarly (75) and $k_0{}^2 = 0$ shows that G_0 satisfies $\nabla \times G_0 = 0$ and $\nabla \cdot G_0 = 0$ in V.

Mixed Boundary Conditions

The sets of functions Ψ and Φ defined in Section II are not the only sets for the expansions of the electric and magnetic fields in a cavity. We can impose the mixed boundary conditions, short and open, upon the functions. Let $S + S'$ be a closed surface (or surfaces) in which we are solving Maxwell's equations. The solutions of

$$\nabla^2\Psi + k^2\Psi = 0$$

$$\left.\begin{aligned} n \times \Psi &= 0 \\ \nabla \cdot \Psi &= 0 \end{aligned}\right\} \text{(on } S\text{)} \qquad \left.\begin{aligned} n \times \nabla \times \Psi &= 0 \\ n \cdot \Psi &= 0 \end{aligned}\right\} \text{(on } S'\text{)} \quad (77)$$

are capable of forming a complete set of orthonormal functions. Similarly, the solutions of

[6] P. M. Morse and H. Feshback, "Methods of Theoretical Physics," McGraw-Hill Book Co., Inc., New York, N. Y., p. 54; 1953.

[7] P. M. Morse and H. Feshback, *loc. cit.*, pp. 738–739. Also see R. Courant and D. Hilbert, "Methods of Mathematical Physics," Interscience Publishers, New York, N. Y., vol. 1, pp. 424–429; 1953.

$$\nabla^2 \boldsymbol{\Phi} + k^2 \boldsymbol{\Phi} = 0$$

$$\left.\begin{aligned} \boldsymbol{n} \times \nabla \times \boldsymbol{\Phi} &= 0 \\ \boldsymbol{n} \cdot \boldsymbol{\Phi} &= 0 \end{aligned}\right\} \text{(on } S) \qquad \left.\begin{aligned} \boldsymbol{n} \times \boldsymbol{\Phi} &= 0 \\ \nabla \cdot \boldsymbol{\Phi} &= 0 \end{aligned}\right\} \text{(on } S') \qquad (78)$$

form another complete set of orthonormal functions.

The $\boldsymbol{\Psi}$'s can be divided into two groups, the $\boldsymbol{E}_a$'s and the $\boldsymbol{F}_\alpha$'s, and the $\boldsymbol{\Phi}$'s into the $\boldsymbol{H}_a$'s and the $\boldsymbol{G}_\lambda$'s. $\boldsymbol{E}_a$ and $\boldsymbol{H}_a$ satisfy the relations

$$k_a \boldsymbol{E}_a = \nabla \times \boldsymbol{H}_a, \qquad k_a \boldsymbol{H}_a = \nabla \times \boldsymbol{E}_a$$

and the boundary conditions

$$\left.\begin{aligned} \boldsymbol{n} \times \boldsymbol{E}_a &= 0 \\ \boldsymbol{n} \cdot \boldsymbol{H}_a &= 0 \end{aligned}\right\} \text{(on } S) \qquad \left.\begin{aligned} \boldsymbol{n} \cdot \boldsymbol{E}_a &= 0 \\ \boldsymbol{n} \times \boldsymbol{H}_a &= 0 \end{aligned}\right\} \text{(on } S').$$

Those are the functions used by Slater. For $\boldsymbol{F}_\alpha$ $(\alpha \neq 0)$, we have

$$k_\alpha \boldsymbol{F}_\alpha = \nabla \psi_\alpha$$

where ψ_α is the solution of

$$\nabla^2 \psi_\alpha + k_\alpha^2 \psi_\alpha = 0$$

$$\psi_\alpha = 0 \ \text{(on } S) \qquad \frac{\partial \psi_\alpha}{\partial n} = 0 \ \text{(on } S').$$

Slater, however, took the boundary condition $\psi_\alpha = 0$ on S and S'. As a result of this fault, he missed in the input impedance a capacitance term which corresponds to the inductance term in (49). In a certain type of cavities, there exists the function $\boldsymbol{F}_0$. The best example may be a cavity with a coupling probe and a coaxial line output as shown in Fig. 7. For $\boldsymbol{G}_\lambda$ $(\lambda \neq 0)$, we have

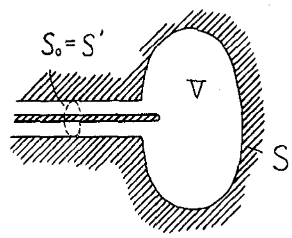


Fig. 7—Cavity with probe.

$$k_\lambda \boldsymbol{G}_\lambda = \nabla \phi_\lambda$$

where ϕ_λ is the solution of

$$\nabla^2 \phi_\lambda + k_\lambda^2 \phi_\lambda = 0$$

$$\phi_\lambda = 0 \ \text{(on } S') \qquad \frac{\partial \phi_\lambda}{\partial n} = 0 \ \text{(on } S).$$

Besides the multiply connected region, we have $\boldsymbol{G}_0$ in the region where S separates S' into two or more than two independent parts. If we take a cross section of the output guide as S' and the cavity wall as S and impose a tangential magnetic field on S' as Slater did, we can easily show that the $\boldsymbol{G}_\lambda$ components of $\boldsymbol{H}$ are all very small and can be neglected. But it does not justify the statement that we need no $\boldsymbol{G}_\lambda$ functions to start with.

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